Supplementary Material
for StainedSweeper: Compact, Variable-Intensity Light-Attenuation Display with Sweeping Tunable Retarders

A DERIVATION PROCESS OF JONES MATRIX OF FOLDED SOLC FILTER

This section describes the process of deriving the Jones matrix in the primary text.

A.1 Derivation Process of Eq. (5)

From Eq. (2),

$$
\begin{aligned}
\mathbf{M}_{+\rho}(\phi) &= \mathbf{R}_\rho^{\mathrm{T}}\mathbf{M}(\phi)\mathbf{R}_\rho \\
&= \begin{bmatrix} \cos\rho & -\sin\rho \\ \sin\rho & \cos\rho \end{bmatrix} \begin{bmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{bmatrix} \begin{bmatrix} \cos\rho & \sin\rho \\ -\sin\rho & \cos\rho \end{bmatrix} \\
&= \begin{bmatrix} e^{-i\phi}\cos^2\rho + e^{i\phi}\sin^2\rho & (e^{-i\phi}-e^{i\phi})\cos\rho\sin\rho \\ (e^{-i\phi}-e^{i\phi})\cos\rho\sin\rho & e^{-i\phi}\sin^2\rho + e^{i\phi}\cos^2\rho \end{bmatrix} \\
&= \begin{bmatrix} e^{-i\phi}\left(\frac{1+\cos2\rho}{2}\right) + e^{i\phi}\left(\frac{1-\cos2\rho}{2}\right) & -i\left(\frac{e^{i\phi}-e^{-i\phi}}{2i}\right)2\cos\rho\sin\rho \\ -i\left(\frac{e^{i\phi}-e^{-i\phi}}{2i}\right)\cdot 2\cos\rho\sin\rho & e^{-i\phi}\left(\frac{1-\cos2\rho}{2}\right) + e^{i\phi}\left(\frac{1+\cos2\rho}{2}\right) \end{bmatrix} \\
&= \begin{bmatrix} \cos\phi - i\cos2\rho\sin\phi & -i\sin2\rho\sin\phi \\ -i\sin2\rho\sin\phi & \cos\phi + i\cos2\rho\sin\phi \end{bmatrix}. \qquad (20)
\end{aligned}
$$

A.2 Derivation Process of Eq. (9)

In this subsection, the derivation of Eq. (9) is explained. First, the derivation based on the Chebyshev polynomial [51] (Eq. (8)) is explained, followed by the derivation by direct computation of the Jones matrix. The latter derivation will be used in the next section when the value of ϕ changes for each retarder.

A.2.1 Derivation Process from Eq. (8)

From Eq. (8) and $N = 4$,

$$
\begin{aligned}
m_{21,4}(\phi) &= -\sin4\rho\sin^2\phi \cdot \frac{\sin2\Lambda}{\sin\Lambda} \\
&= -2\sin4\rho\sin^2\phi \cdot \cos\Lambda \\
&= -2\sin4\rho\sin^2\phi \cdot \cos(\cos^{-1}(\cos^2\phi - \sin^2\phi\cos4\rho)) \\
&= -2\sin4\rho\sin^2\phi(\cos^2\phi - \sin^2\phi\cos4\rho). \qquad (21)
\end{aligned}
$$

A.2.2 Derivation Process from direct calculation of Jones Matrix

From Eq. (6),

$$
\begin{aligned}
\mathbf{M}_{-\rho}(\phi)\mathbf{M}_{+\rho}(\phi)\mathbf{M}_{-\rho}(\phi)\mathbf{M}_{+\rho}(\phi) &= \begin{bmatrix} m_{11,4}(\phi) & m_{12,4}(\phi) \\ m_{21,4}(\phi) & m_{22,4}(\phi) \end{bmatrix} \\
&= \begin{bmatrix} m_{11,2}(\phi) & m_{12,2}(\phi) \\ m_{21,2}(\phi) & m_{22,2}(\phi) \end{bmatrix} \begin{bmatrix} m_{11,2}(\phi) & m_{12,2}(\phi) \\ m_{21,2}(\phi) & m_{22,2}(\phi) \end{bmatrix}.
\end{aligned}
$$

From the equation, we derive

$$
m_{21,4}(\phi) = m_{21,2}(\phi)(m_{11,2}(\phi) + m_{22,2}(\phi)). \qquad (22)
$$

From Eq. (20), $\mathbf{M}_{-\rho}(\phi)$ is denoted as

$$
\mathbf{M}_{-\rho}(\phi) = \begin{bmatrix} \cos\phi - i\cos2\rho\sin\phi & i\sin2\rho\sin\phi \\ i\sin2\rho\sin\phi & \cos\phi + i\cos2\rho\sin\phi \end{bmatrix}.
$$

Thus, the Jones matrix of the Solc filter for $N = 2$ is

$$
\begin{aligned}
\mathbf{M}_{-\rho}(\phi)\mathbf{M}_{+\rho}(\phi) &= \begin{bmatrix} \cos\phi - i\cos2\rho\sin\phi & -i\sin2\rho\sin\phi \\ -i\sin2\rho\sin\phi & \cos\phi + i\cos2\rho\sin\phi \end{bmatrix} \begin{bmatrix} \cos\phi - i\cos2\rho\sin\phi & i\sin2\rho\sin\phi \\ i\sin2\rho\sin\phi & \cos\phi + i\cos2\rho\sin\phi \end{bmatrix} \\
&= \begin{bmatrix} m_{11,2}(\phi) & m_{12,2}(\phi) \\ m_{21,2}(\phi) & m_{22,2}(\phi) \end{bmatrix}. \qquad (23)
\end{aligned}
$$

Each element of Eq. (23) is calculated separately as

$$\begin{aligned} m_{11,2}(\phi) &= (\cos\phi - i\cos 2\rho\sin\phi)^2 + \sin^2 2\rho\sin^2\phi, \\ m_{22,2}(\phi) &= (\cos\phi + i\cos 2\rho\sin\phi)^2 + \sin^2 2\rho\sin^2\phi, \end{aligned}$$

and

$$\begin{aligned} m_{21,2}(\phi) &= (\cos\phi - i\cos 2\rho\sin\phi)(-i\sin 2\rho\sin\phi) + i\sin 2\rho\sin\phi(\cos\phi + i\cos 2\rho\sin\phi) \\ &= -2\sin 2\rho\cos 2\rho\sin^2\phi \\ &= -\sin 4\rho\sin^2\phi. \end{aligned} \tag{24}$$

By substituting these into Eq. (23), we obtain

$$\begin{aligned} m_{21,4}(\phi) &= -\sin 4\rho\sin^2\phi(2\cos^2\phi - 2\cos^2 2\rho\sin^2\phi + 2\sin^2 2\rho\sin^2\phi) \\ &= -2\sin 4\rho\sin^2\phi(\cos^2\phi - \sin^2\phi(\cos^2 2\rho - \sin^2 2\rho)) \\ &= -2\sin 4\rho\sin^2\phi(\cos^2\phi - \sin^2\phi\cos 4\rho). \end{aligned}$$

B DERIVATION PROCESS OF JONES MATRIX OF STAINEDSWEEPER

This section describes the derivation process of the Jones matrix for a folded Solc filter using a waveplate with different retardation, as used in our StainedSweeper. From Eq. (11), the Jones matrix of the reflective Solc filter with different waveplates is

$$\begin{aligned} \mathbf{M}_{\mathrm{Refl},4} &= (\mathbf{J}\mathbf{M}^{\mathrm{T}}_{+\rho}(\phi_1)\mathbf{J})(\mathbf{J}\mathbf{M}^{\mathrm{T}}_{-\rho}(\phi_2)\mathbf{J})\cdot(\mathbf{J}\mathbf{Q}^{\mathrm{T}}\mathbf{J})\cdot\mathbf{J}\cdot\mathbf{Q}\cdot\mathbf{M}_{-\rho}(\phi_2)\mathbf{M}_{+\rho}(\phi_1) \\ &= \mathbf{M}_{-\rho}(\phi_1)\mathbf{M}_{+\rho}(\phi_2)\mathbf{M}_{-\rho}(\phi_2)\mathbf{M}_{+\rho}(\phi_1) \\ &= \begin{bmatrix} m_{11,4}(\phi_1,\phi_2) & m_{12,4}(\phi_1,\phi_2) \\ m_{21,4}(\phi_1,\phi_2) & m_{22,4}(\phi_1,\phi_2) \end{bmatrix}, \end{aligned} \tag{25}$$

where $m_{ij,N}(\phi_1,\phi_2)$ is each element of the Solc filter when combining waveplates with different retardation ϕ_1 and ϕ_2. From Eq. (23),

$$\begin{aligned} \mathbf{M}_{-\rho}(\phi_2)\mathbf{M}_{+\rho}(\phi_1) &= \begin{bmatrix} \cos\phi_2 - i\cos 2\rho\sin\phi_2 & -i\sin 2\rho\sin\phi_2 \\ -i\sin 2\rho\sin\phi_2 & \cos\phi_2 + i\cos 2\rho\sin\phi_2 \end{bmatrix} \begin{bmatrix} \cos\phi_1 - i\cos 2\rho\sin\phi_1 & i\sin 2\rho\sin\phi_1 \\ i\sin 2\rho\sin\phi_1 & \cos\phi_1 + i\cos 2\rho\sin\phi_1 \end{bmatrix} \\ &= \begin{bmatrix} m_{11,2}(\phi_1,\phi_2) & m_{12,2}(\phi_1,\phi_2) \\ m_{21,2}(\phi_1,\phi_2) & m_{22,2}(\phi_1,\phi_2) \end{bmatrix}. \end{aligned} \tag{26}$$

Similarly,

$$\begin{aligned} \mathbf{M}_{-\rho}(\phi_1)\mathbf{M}_{+\rho}(\phi_2) &= \begin{bmatrix} \cos\phi_1 - i\cos 2\rho\sin\phi_1 & -i\sin 2\rho\sin\phi_1 \\ -i\sin 2\rho\sin\phi_1 & \cos\phi_1 + i\cos 2\rho\sin\phi_1 \end{bmatrix} \begin{bmatrix} \cos\phi_2 - i\cos 2\rho\sin\phi_2 & i\sin 2\rho\sin\phi_2 \\ i\sin 2\rho\sin\phi_2 & \cos\phi_2 + i\cos 2\rho\sin\phi_2 \end{bmatrix} \\ &= \begin{bmatrix} m_{11,2}(\phi_2,\phi_1) & m_{12,2}(\phi_2,\phi_1) \\ m_{21,2}(\phi_2,\phi_1) & m_{22,2}(\phi_2,\phi_1) \end{bmatrix}. \end{aligned} \tag{27}$$

From the equations above, $m_{11,2}(\phi_2,\phi_1) = m_{11,2}(\phi_1,\phi_2)$ and $m_{22,2}(\phi_2,\phi_1) = m_{22,2}(\phi_1,\phi_2)$. Thus, by substituting Eq. (26) and Eq. (27) into Eq. (25), we obtain

$$m_{21,4}(\phi_1,\phi_2) = m_{11,2}(\phi_1,\phi_2)m_{21,2}(\phi_2,\phi_1) + m_{22,2}(\phi_1,\phi_2)m_{21,2}(\phi_1,\phi_2). \tag{28}$$

From Eq. (26), we obtain

$$\begin{aligned} m_{21,2}(\phi_1,\phi_2) &= (\cos\phi_1 - i\cos 2\rho\sin\phi_1)(-i\sin 2\rho\sin\phi_2) + i\sin 2\rho\sin\phi_1(\cos\phi_2 + i\cos 2\rho\sin\phi_2) \\ &= i\sin 2\rho(\sin\phi_1\cos\phi_2 - \cos\phi_1\sin\phi_2) - 2\sin 2\rho\cos 2\rho\sin\phi_1\sin\phi_2 \\ &= i\sin 2\rho\sin(\phi_1 - \phi_2) - \sin 4\rho\sin\phi_1\sin\phi_2 \\ &= iA - B, \end{aligned} \tag{29}$$

$$\begin{aligned} m_{21,2}(\phi_2,\phi_1) &= i\sin 2\rho\sin(\phi_2 - \phi_1) - \sin 4\rho\sin\phi_2\sin\phi_1 \\ &= -iA - B, \end{aligned} \tag{30}$$

$$\begin{aligned} m_{11,2}(\phi_1,\phi_2) &= (\cos\phi_1 - i\cos 2\rho\sin\phi_1)(\cos\phi_2 - i\cos 2\rho\sin\phi_2) + \sin^2 2\rho\sin\phi_1\sin\phi_2 \\ &= \cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2\cos 4\rho - i\cos 2\rho\sin(\phi_1 + \phi_2) \\ &= C - iD, \end{aligned} \tag{31}$$

$$\begin{aligned} m_{22,2}(\phi_1,\phi_2) &= (\cos\phi_1 + i\cos 2\rho\sin\phi_1)(\cos\phi_2 + i\cos 2\rho\sin\phi_2) + \sin^2 2\rho\sin\phi_1\sin\phi_2 \\ &= C + iD, \end{aligned} \tag{32}$$

where $A = \sin 2\rho\sin(\phi_1 - \phi_2)$, $B = \sin 4\rho\sin\phi_1\sin\phi_2$, $C = \cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2\cos 4\rho$ and $D = \cos 2\rho\sin(\phi_1 + \phi_2)$.

Finally, substituting the equations above into Eq. (28), we obtain

$$
\begin{aligned}
m_{21,4}(\phi_1,\phi_2) &= (-iA-B)(C-iD)+(iA-B)(C+iD) \\
&= -2BC-2AD \\
&= -2\sin 4\rho \sin\phi_1 \sin\phi_2(\cos\phi_1\cos\phi_2-\sin\phi_1\sin\phi_2\cos 4\rho)-2\sin 2\rho\cos 2\rho\sin(\phi_1-\phi_2)\sin(\phi_1+\phi_2) \\
&= -2\sin 4\rho \sin\phi_1 \sin\phi_2(\cos\phi_1\cos\phi_2-\sin\phi_1\sin\phi_2\cos 4\rho)-\sin 4\rho\sin(\phi_1-\phi_2)\sin(\phi_1+\phi_2). \qquad (33)
\end{aligned}
$$

In this equation, if we set $\phi_1=\phi_2=\phi$, it can be verified that it is consistent with Eq. (9). When $\rho=\pi/16$, Eq. (17) is derived as

$$
m_{21,4}(\phi_1,\phi_2) = -\sin\phi_1\sin\phi_2(\sqrt{2}\cos\phi_1\cos\phi_2-\sin\phi_1\sin\phi_2)-\frac{1}{\sqrt{2}}\sin(\phi_1-\phi_2)\sin(\phi_1+\phi_2).
$$